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RESEARCH ARTICLE

# Fair (intra-bank transfer) prices for credits with stochastic recovery

**Johannes Leitner**

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**Abstract** In a discrete time stationary setting we derive explicit formulas for the value of a coupon paying defaultable credit contract with stochastic recovery. The pricing method is based on a perfect replication argument using simple 1-period insurance contracts.

**Keywords** Defaultable term structure · Stochastic recovery · CDS · Intra-bank transfer prices

**JEL Classification Numbers** G12 · G22

## 1 Introduction

We consider a bank consisting of a central unit and its local branches. We assume that each branch is treated as a profit centre, but has a comparatively small number of customers and does not own substantial risk capital. Therefore, the stand-alone portfolio of each branch does not provide enough diversification in order to be able to offer its costumers a competitive interest rate. This can only be achieved at the global bank level. We are looking for a fair method to transfer the credits from the branches to the central unit, where enough risk

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J. Leitner (✉)
Institute for Mathematical Economics, Vienna University of Technology,
Wiedner Hauptstraße 8-10/105-1, 1040 Vienna, Austria
e-mail: jleitner@fam.tuwien.ac.at

capital is available, the diversification effect for the total credit portfolio is much better and e.g. securitization deals can efficiently be engineered. One way to do this is to let the branches still receive the coupon and terminal payments from their customers, and let the branches buy insurance against default (and refinancing) from the central unit. We are going to demonstrate in a discrete time stationary setting, using simple 1-period insurance contracts or 1-period credit default swaps (CDS), that this can be done in such a way, that each stand-alone credit becomes completely risk-free for the responsible branch. Our aim here is not to develop a general framework for risk-free valuation of cash-flows relative to the prices for 1-period insurance contracts, but to derive (under a weak additional condition) explicit formulas describing the relation between initial value of the credit contract, coupon, terminal payment, default probability and stochastic recovery fraction. In particular, we find an explicit formula for the defaultable term structure (with and without coupon payments).

In principle, the approach we are proposing can be used for pricing of defaultable bonds in the market and not only for intra-bank valuation. However, we think that for intra-bank transfer prices it is a lot easier to implement such an approach since in some sense the intra-bank cash-flows are virtual, so no real money is at stake. The bank's central unit can control its risk appetite and the profitability of its branches by choosing appropriate 1-period insurance prices. Furthermore, these prices result into incentives for the branches to charge the customer interest rates which reflect their default risk and the recovery distribution.

Our stationary setting works best for small private customer credits, which can be assumed to be only weakly correlated, and where after an initial credit ranking no further information about the costumer is collected. For a continuous time version of this approach and for further economic motivation see Leitner (2006).

## 2 The model

Let us start by introducing a simple stationary model for a coupon paying defaultable credit with random recovery. Consider the following credit contract $C$, specified by the payments $(x_i)_{0\leq i\leq T}, 0 < T < \infty$, from the lender's point of view: $x_0 = -B + c, x_i = c$ for $1 \leq i \leq T-1$ and $x_T = M$, i.e. at initial time $t = 0$ the amount $B$ is handed out to the borrower in return for coupon payments $c$ at times $t = 0, \dots, T-1$, and a final payment of $M$ at maturity time $T$. We assume $B - c \geq 0$ and $c, M \geq 0$. However, we allow the borrower to default with a certain probability, in which case coupon payments stop and instead only the amount $XM$ is received by the lender at default time (e.g. by selling a security given by the borrower) and no further payments are made, where $X$ is a $[0, 1]$-valued random variable independent of the time $\tau$ of default, modelling the so-called recovery fraction. The stopping time $\tau$ of default is assumed to take values in $\{1, \dots, T\} \cup \{\infty\}$ where $\{\tau = \infty\}$ is the event of no default. The discrete time filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ we are working with is the one generated by the process $X\mathbf{1}_{[\tau,T]}$ and we assume $P(\tau = t+1|\tau > t) = p \in [0,1]$ for all $0 \le t < T$, i.e. the model is stationary in the sense that at time $0 \le t < \tau \wedge T$ given the information $\mathcal{F}_t$, the conditional probability that default occurs at time $t+1$ is constant.

Next we are going to describe the accumulated cash flow of the lender: Assume a constant 1-period risk-free interest rate $r \ge 0$ to be given and set $R = 1 + r \ge 1$, so that $R^{-1}$ is the price of a risk-free 1-period zero coupon bond at each time $0 \le t < T$.

The accumulated cash flow $(Z_t^C)_{0 \le t \le T}$ of the lender generated by the credit contract $C$ (and refinancing) is given by $Z_0^C = -B + c$ and for $1 \le t < \tau \wedge T$ or $t = \tau = T$ by

$$Z_t^C = Z_{t-1}^C R + c\mathbf{1}_{\{t<\tau\}} + XM\mathbf{1}_{\{\tau=t\}}, \tag{1}$$

$Z_T^C = Z_{T-1}^C R + M$ on $\{\tau = \infty\}$ and $Z_t^C = Z_\tau^C R^{t-\tau}$ for $\tau < t \le T$.

## 2.1 Single-period credit insurance contracts

Next we are going to consider a 1-period credit insurance contract or credit default swap. We assume that the lender can buy (short-term) insurance against default in the following way: In return for an insurance premium $Y_t = f_t(I_{t+1}\mathbf{1}_{\{\tau=t+1\}})$ paid at time $0 \le t < \tau \wedge T$, the insured receives the $\mathcal{F}_{t+1}$-measurable random amount $I_{t+1} \ge 0$ at time $t+1$ on $\{\tau = t+1\}$, i.e. if default occurs at time $t+1$. Here $f_t$ denotes a functional that assigns an $\mathcal{F}_t$-measurable random number to any non-negative bounded random variable $I$. We always assume that $f_t(I)$ depends only on the distribution of $I$ given $\mathcal{F}_t$. To be precise, $f_t(I)$ is calculated point-wise from the conditional distribution function $F_{I|\mathcal{F}_t} := P(I \le \cdot|\mathcal{F}_t)$ of $I \in L_+^\infty$, given $\mathcal{F}_t$, which is a $\mathcal{F}_t$-measurable function assuming its values in the set of distribution functions. However, suppressing this conditional dependency we write $f_t(I)$ instead of e.g. $f(I|\,\mathcal{F}_t)$.

Let us fix in addition the following minimal assumptions on $f_t, t \ge 0$, we assume to hold throughout:

1. Vanishing no loss premium: $f_t(0) = 0$.
2. Sure loss additivity: $f_t(I + k) = f_t(I) + R^{-1}k$ for all $k \ge 0$ and all $I \in L_+^\infty$.
3. Strict monotonicity: $f_t(I) \le f_t(\tilde{I})$ for $I \le \tilde{I}$, and $f_t(I) < f_t(\tilde{I})$ for $I \le \tilde{I}$ and not a.s. $I = \tilde{I}$, for all $I, \tilde{I} \in L_+^\infty$.

The accumulated cash flow $(Z_t^I)_{0 \le t \le T}$ generated by such a stream of 1-period insurance contracts is given by $Z_0^I = -Y_0$ and for $1 \le t < \tau \wedge T$ or $t = \tau = T$ by

$$Z_t^I = Z_{t-1}^I R - Y_t\mathbf{1}_{\{t<\tau\}} + I_t\mathbf{1}_{\{\tau=t\}}, \tag{2}$$

$Z_T^I = Z_{T-1}^I R$ on $\{\tau = \infty\}$ and $Z_t^I = Z_\tau^I R^{t-\tau}$ for $\tau < t \le T$.

We find for the total accumulated cash flow $Z := Z^C + Z^I$, $Z_0 = -B + c - Y_0$ and for $1 \le t < \tau \wedge T$ or $t = \tau = T$

$$Z_t = Z_{t-1}R + (c - Y_t)\mathbf{1}_{\{t<\tau\}} + (XM + I_t)\mathbf{1}_{\{\tau=t\}}, \tag{3}$$

$Z_T = Z_{T-1}R + M$ on $\{\tau = \infty\}$ and $Z_t = Z_\tau R^{t-\tau}$ for $\tau < t \le T$.

The trick of our approach is now *not* to insure e.g. the loss given default $(1 - X)M$, but to choose $I_t$ such that $Z_t = 0$ if default occurs at time $t$. For $p \in (0, 1]$, this can only be achieved by defining for $0 < t < \tau \wedge T$

$$I_t := -XM - Z_{t-1}R = (1 - X)M - (MR^{-1} + Z_{t-1})R, \tag{4}$$

resulting into $Z_t = 0$ for all $\tau \le t \le T$. However insurance contracts exist only for $I_t \ge 0$, which is equivalent to the following condition:

$$Z_t \le -\bar{x}MR^{-1}, \quad 0 \le t < \tau \wedge T, \tag{5}$$

where $\bar{x} := \mathit{esssup}\, X \in [0, 1]$. We can also extend the functionals $f_t$ in a monotonic way to $L^\infty$.

The objective is now to choose the constants $c, B, M$ in such a way that in the case of no default $Z_T = 0$ holds too! This means that no matter whether default occurred or not, at time $\tau \wedge T$ the accumulated cash flow vanishes, i.e. the lender has exactly the same wealth as at initial time $t = 0$ and as in the case of not having entered the credit contract in the first place.

In this situation $V_0 := B - c$ can be interpreted as the fair price at time $t = 0$ of the defaultable coupon plus terminal payment cash stream generated by the credit contract $C$ (excluding the first coupon payment of $c$ at time $t = 0$). Similarly, at $0 < t < \tau \wedge T$, $-Z_{t-1}R + c$ can be interpreted as the value of the remaining cash flow generated by the credit contract $C$ after time $t$ (before having paid $Y_t$ for insurance). Knowing this value is very convenient if the customer wishes to change the terms of the credit contract. For $\tilde{c}$ and $\tilde{M}$ such that the corresponding initial price $\tilde{B} - \tilde{c}$ calculated for time $t$ equals $V_t := -Z_{t-1}R - c$ (plus fee) the bank can change the contract cost neutral.

Since the credit is now completely risk-free for the branch and disappears from the branch's books at $\tau \wedge T$, the branch can just charge the costumer at initial time a fee covering its administration costs (plus a profit) or add a running costs fee in the coupon payments $c$. This is also important from a risk management point of view. It is difficult for a stand-alone branch to do a proper risk management, in particular since the branch itself does not own (substantial) risk capital. If on the other hand branches are to be operated as profit centres, life becomes a lot easier if the branches are in a completely risk-free state, and all the risk is taken by the central unit, which is properly equipped to deal with it. We are working here in a simplified stationary setting. For applications it is important to note, that even if default probabilities change over time, the branch's position remains completely risk-free, the risk of unfavourable changes (but also the chance of improvements of credit ranking) lies completely with the central unit.

Since $Y_t$ depends on $Z_{t+1}$, it is in general not easy to solve this problem explicitly. Before doing this under an additional condition on $f_t, 0 \le t$, we take a look at the single period case.

## 2.2 Defaultable single-period bond prices

Consider a defaultable 1-period bond, i.e. $T = 1$. Set $\bar{f} := \bar{f}(X, M, p) := f_0((1 - X)M\mathbf{1}_{\{\tau=1\}})$.

Assume first $0 \le p < 1$. The condition $Z_1 = 0$ implies $(-B + c - Y_0)R = -M$ for $\tau = \infty$, hence $I_1 = -XM + (-B + c - Y_0)R = (1 - X)M \ge 0$ on $\{\tau = 1\}$ and

$$Y_0 = f_0(I_1\mathbf{1}_{\{\tau=1\}}) = f_0((1 - X)M\mathbf{1}_{\{\tau=1\}}) = \bar{f}.$$

As it is intuitively clear, we just have to insure the loss given default $(1 - X)M$ and we find

$$B = MR^{-1} + c - \bar{f}. \tag{6}$$

and thus $B - c = MR^{-1} - \bar{f} \ge 0$.

Note that for $P(X = 1) > 0$ the cash-flow on $\{\tau = 1, X = 1\}$ equals the one on $\{\tau = \infty\}$ since $I_1 = 0$, i.e. the credit contract is for $T = 1$ indistinguishable from a credit contract with default time $\tilde{\tau}$ such that $\{\tau = 1, X < 1\} \subseteq \{\tilde{\tau} = 1\} \subseteq \{\tau = 1\}$ and recovery fraction $X$. Since then $f_0((1-X)\mathbf{1}_{\{\tilde{\tau}=1\}}) = f_0((1-X)\mathbf{1}_{\{\tau=1\}})$, we find the corresponding initial prices and insurance premia to coincide as it should be.

For $p = 1$ we have $I_1 \ge 0$ iff $Y_0 \ge \bar{x}MR^{-1} - B + c$ and using sure loss additivity we then find

$$\begin{aligned} Y_0 &= f_0(I_1\mathbf{1}_{\{\tau=1\}}) = f_0((1 - X)M - (MR^{-1} - B + c - Y_0)R) \\ &= -(MR^{-1} - B + c - Y_0) + \bar{f} \end{aligned}$$

implying again (6). Note that $Y_0$ is in this case not uniquely determined and any $Y_0 \ge \bar{x}MR^{-1} - B + c$ results into $Z_1 = 0$.

Note also that strict monotonicity implies $B - c = MR^{-1} - \bar{f} \ge 0$ and $B = c$ iff $M = 0$ or $M > 0, \bar{x} = 0, p = 1$, i.e. the contract generates a zero cash flow.

### *2.2.1 Sub-additivity*

Assume that the premium calculation principle satisfies in addition the following sub-additivity property: $f_0(I + \tilde{I}) \le f_0(I) + f_0(\tilde{I})$ for all $I, \tilde{I} \in L^\infty_+$ (or $L^\infty$) . Note that convexity and positive homogeneity of $f_0$ imply the sub-additivity property. For credits $C_1, C_2, C$, with initial prices $B_1, B_2, B$, calculated from coupon payments $c_1, c_2 \ge 0, c := c_1 + c_2$, terminal payments $M_1, M_2 \ge 0, M := M_1 + M_2$, default events $\{\tau_1 = 1\}, \{\tau_2 = 1\}, \{\tau = 1\} := \{\tau_1 = 1\} \cup \{\tau_2 = 1\}$ and recovery ratios $X_1, X_2, X := \frac{(1-X_1)M_1\mathbf{1}_{\{\tau_1=1\}}+(1-X_2)M_2\mathbf{1}_{\{\tau_2=1\}}}{M}$, $\left(\frac{0}{0} := 0\right)$, respectively, sub-additivity implies for the corresponding insurance payments $\bar{f}_1, \bar{f}_2, \bar{f}$, that $\bar{f}_1 + \bar{f}_2 \geq \bar{f}$ holds, i.e. the insurer (central unit) gains $\bar{f}_1 + \bar{f}_2 - \bar{f}$ by separately insuring the two credits $C_1, C_2$ instead of insuring the combined credit $C$.

#### *2.2.2 General terminal values*

From this single period consideration it is clear that, at least for a finite probability space, e.g. modelling a finite state Markov chain for the risk-free 1-period interest rate and the costumers credit rating, we can replicate arbitrary terminal values, step by step backwards recursively determining the necessary capital and single-period insurances (with claims in $L^\infty$) until finding the initial value. If sub-additivity holds for all $f_t$, then sub-additivity carries over to the initial value. However, even for terminal values as simple as the one of the above credit contract it is in general difficult to find explicit formulas. By introducing an additional assumption on the functionals $f_t$, the problem becomes at least for our defaultable credit tractable.

### 2.3 A tractable premium calculation principle

Next, we introduce the following assumption on $f_t, t \geq 0$: We assume that for a continuous strictly increasing function $\alpha$ and setting $\bar{\alpha} := \alpha(p)$,

$$f_t((I+k)\mathbf{1}_{\{\tau=t+1\}}) = f_t(I\mathbf{1}_{\{\tau=t+1\}}) + \bar{\alpha}R^{-1}k, \tag{7}$$

holds for all $0 \leq t < \tau \wedge T$, constants $k \geq 0$ and $I \in L^\infty_+$. Note that the vanishing no loss premium implies $\alpha(0) = 0$ and sure loss additivity implies $\alpha(1) = 1$.

In particular, (7) holds for $\bar{\alpha} = f_t(\mathbf{1}_{\{\tau=t+1\}})$, if $f_t$ is co-monotonic additive [see e.g. Denneberg (1994) for the notions of co-monotonicity and co-monotonic additivity], which implies positive homogeneity, and if $f_t(\mathbf{1}_D)$ is continuous in $P(D)$ for $D \in \mathcal{F}_{t+1}$: $f_t((I+k)\mathbf{1}_{\{\tau=t+1\}}) = f_t(I\mathbf{1}_{\{\tau=t+1\}}) + kf_t(\mathbf{1}_{\{\tau=t+1\}})$. Such type of functionals for calculating insurance premia are well-known in insurance mathematics, see e.g. Wang (1996), Wang et al. (1997) and Goovaerts et al. (2003).

A simple example is the following co-monotonic additive functional:

$$f_t(I) := R^{-1}\int_0^1 F^{-1}_{I|\mathcal{F}_t}(u)h(u)du \tag{8}$$

where $h : (0,1) \to (0,\infty)$ satisfies $\int_0^1 h(u)du = 1$ and $F^{-1}_{I|\mathcal{F}_t}$ denotes a generalized inverse of the conditional cdf $F_{I|\mathcal{F}_t}$ of $I$ given $\mathcal{F}_t$, e.g. $F^{-1}_{I|\mathcal{F}_t}(u) := \inf\{s \in$

$\mathbf{R}|F_{I|\mathcal{F}_t}(s) > u\}, u \in [0,1]$. It is easy to check, that in this case $f_t$ satisfies condition (7) for $\alpha(p) = \int_{1-p}^{1} h(u)du$ and strict monotonicity. Furthermore, note that for this example we also have positive homogeneity, i.e. $f_t(kI) = kf_t(I)$ for all $k \geq 0$, and sure loss additivity. Clearly, $f_t$ can be extended to all of $L^\infty$ and it is also well-known, that $f_t$ is convex iff $h$ is in addition non-decreasing, see Acerbi (2002).

## 2.4 Defaultable multi-period bond prices

Assuming now condition (7) we are going to solve the multi-period case $1 < T < \infty$ explicitly. Assuming condition (5) to hold, we find for the total accumulated cash flow for $1 \leq t < \tau \wedge T$

$$\begin{aligned} Z_t &= Z_{t-1}R + c - f_t(I_{t+1}\mathbf{1}_{\{\tau=t+1\}}) \\ &= Z_{t-1}R + c - f_t(((1-X)M - (Z_tR + M))\mathbf{1}_{\{\tau=t+1\}}) \\ &= Z_{t-1}R + c + (Z_t + MR^{-1})\bar{\alpha} - \bar{f}, \end{aligned} \tag{9}$$

where we have used the fact that on $\{t < \tau\}$ the distribution of $X$ and $\tau$ at time $t$, given $\mathcal{F}_t$, does not depend on $t$.

We consider the cases $p = 0, 0 < p < 1, p = 1$, separately:

### *2.4.1 The case* $0 < p < 1$

Since $0 < \bar{\alpha} < 1$, solving Eq. (9) for $Z_t$, we find $(Z_t)_{0 \leq t \leq T}$ to solve the following recursive equation for $1 \leq t < \tau \wedge T$:

$$z_t = z_{t-1}\beta + \gamma, \tag{10}$$

where $\beta := \frac{R}{1-\bar{\alpha}} > 1$ and $\gamma := \frac{c + \bar{\alpha}MR^{-1} - \bar{f}}{1-\bar{\alpha}}$ are constants. We have for $1 < t \leq T$

$$z_{t-1} = z_0\beta^{t-1} + \gamma\sum_{i=1}^{t-1}\beta^{i-1} = z_0\beta^{t-1} + \gamma\frac{\beta^{t-1} - 1}{\beta - 1}. \tag{11}$$

The terminal condition $Z_T = 0$ on $\{\tau = \infty\}$ is equivalent to $Z_{T-1} = -MR^{-1}$ and we find, solving with $t = T$ Eq. (11) for $z_0$,

$$Z_0 = \frac{-MR^{-1} - \gamma\frac{\beta^{T-1}-1}{\beta-1}}{\beta^{T-1}}. \tag{12}$$

Plugging this expression into (11) we find for $1 \le t \le \tau \wedge T$:

$$\begin{aligned} Z_{t-1} &= \left(-MR^{-1} - \gamma\frac{\beta^{T-1}-1}{\beta-1}\right)\beta^{t-T} + \gamma\frac{\beta^{t-1}-1}{\beta-1} \\ &= -MR^{-1}\beta^{t-T} + \gamma\frac{\beta^{t-T}-1}{\beta-1} \\ &= -MR^{-1} + R^{-1}(rMR^{-1} - c + \bar{f})\Phi_{T-t}(\beta^{-1}), \end{aligned} \tag{13}$$

where we define for $s \ge 0$ and $y \in [0, 1)$, $(0^0 := 1)$,

$$\Phi_s(y) := \frac{1-y^s}{1-y}. \tag{14}$$

Note that $(Z_t)_{0\le t\le T}$ is strictly increasing on $0 \le t < \tau \wedge T$ iff $c > rMR^{-1} + \bar{f}$ and constant iff equality holds, since $\Phi_{T-t}(\beta^{-1})$ is strictly decreasing in $1 \le t \le T$.

Next we give a condition equivalent to condition (5). From (13) we find (5) to hold iff for all $1 \le t \le \tau \wedge T$

$$RZ_{t-1} + \bar{x}M = (\bar{x}-1)M + (rMR^{-1} - c + \bar{f})\Phi_{T-t}(\beta^{-1}) \le 0,$$

or equivalently iff $c \ge 0$ satisfies

$$c \ge rMR^{-1} + \bar{f} - \frac{(1-\bar{x})M}{\Phi_{T-1}(\beta^{-1})} =: c_0. \tag{15}$$

That is with $\delta := c - c_0$, condition (5) holds iff $\delta \ge \max\{-c_0, 0\}$. We assume from now on always $\delta \ge \max\{-c_0, 0\}$. Then (13) becomes

$$Z_{t-1} = -MR^{-1} + R^{-1}(1-\bar{x})M\frac{1-\beta^{t-T}}{1-\beta^{1-T}} - \delta R^{-1}\Phi_{T-t}(\beta^{-1}), \tag{16}$$

For $\bar{x} = 1$ we have $\delta = 0$ iff $Z_t = MR^{-1}$ for all $0 \le t < \tau \wedge T$. Note also that for $\delta = 0$ we then have $Y_t = \bar{f}$ for all $0 \le t < \tau \wedge T$, in which case the stream of 1-period insurance contracts generates the same cash-flow as a classical credit default swap. See e.g. Bielecki and Rutkowski (2002) for material on CDS.

For the insurance payments we find for $0 \le t < \tau \wedge T$

$$I_{t+1} = (1-X)M + \delta\Phi_{T-t-1}(\beta^{-1}) - (1-\bar{x})M\frac{1-\beta^{t+1-T}}{1-\beta^{1-T}}, \tag{17}$$

hence for the corresponding premia

$$Y_t = \bar{f} + \bar{\alpha}R^{-1}\left(\delta\Phi_{T-t-1}(\beta^{-1}) - (1-\bar{x})M\frac{1-\beta^{t+1-T}}{1-\beta^{1-T}}\right), \tag{18}$$

the interpretation being that we do not only have to insure the loss given default $(1-X)M$, which would only result into a premium of $\bar{f}$, but also against the risk of losing the remaining coupon payments!

For the net income of the branch at time $0 \leq t < \tau \wedge T$ we have

$$c - Y_t = rMR^{-1} + \delta \frac{r + \bar{\alpha}\beta^{t+1-T}}{r + \bar{\alpha}} - ((1-\bar{x})MR^{-1}) \frac{r + \bar{\alpha}\beta^{t+1-T}}{1 - \beta^{1-T}}. \tag{19}$$

Next we calculate the initial bond price $B$. Firstly, $Z_0 = -B + c - Y_0 = -B + c - f(-XM - Z_0 R) = -B + c + \bar{\alpha}(MR^{-1} + Z_0) - \bar{f}$ implies

$$Z_0 = \frac{-B + c + \bar{\alpha}MR^{-1} - \bar{f}}{1 - \bar{\alpha}}. \tag{20}$$

Equations (12) and (20) lead to the following relation between $B, c, M$: We find

$$\begin{aligned} B &= \frac{M}{\beta^T} + (c + \bar{\alpha}MR^{-1} - \bar{f})\Phi_T(\beta^{-1}) \\ &= \frac{M}{\beta^T} + \Big((r + \bar{\alpha})MR^{-1} + \delta\Big)\, \Phi_T(\beta^{-1}) - (1-\bar{x})M \frac{1 - \beta^{-T}}{1 - \beta^{1-T}}. \end{aligned} \tag{21}$$

Note that (21) coincides with Eq. (6) for $T = 1$.

From the above formulas it is now easy to calculate for $t = 0$

$$\begin{aligned} -B + c &= Z_0 + Y_0 \\ &= (1-\bar{x})MR^{-1}(1-\bar{\alpha}) - MR{-}1 + \bar{f} - \delta R^{-1}(1-\bar{\alpha})\Phi_{T-1}(\beta^{-1}) \\ &= f_0((\bar{x} - X)M) - \bar{x}MR^{-1} - \delta R^{-1}(1-\bar{\alpha})\Phi_{T-1}(\beta^{-1}). \end{aligned}$$

$f_0((\bar{x} - X)M) - \bar{x}MR^{-1} \leq -\bar{x}MR^{-1}(1-\bar{\alpha})$ implies $-B + c \leq 0$, i.e. the branch is not borrowing from the costumer. Furthermore, $f_0((\bar{x} - X)M) - \bar{x}MR^{-1} = 0$ implies $M = 0$ or $M > 0, \bar{x} = X = 0$. For $M = 0$ we have $c_0 = 0$, hence $c = \delta \geq 0$ and $-B + c = 0$ implies $c = 0$, i.e. the credit contract generates the zero cash-flow for all involved parties. For the second case, $M > 0, \bar{x} = X = 0$, we have $c_0 = (r + \bar{\alpha})MR^{-1} - \frac{M}{\Phi_{T-1}(\beta^{-1})} = M(1 - \beta^{-1})\left(1 - \frac{1}{1-\beta^{1-T}}\right) < 0$, hence $\delta > 0$. Therefore $-B + c < 0$ if $M > 0, \bar{x} = X = 0$. Note that this is a kind of no-arbitrage property.

*2.4.2 The case $p = 0$*

Since we are then in the risk-free situation, we easily find Eq. (21) to hold for $p = 0$ and $1 \leq T < \infty$ if we set

$$\Phi_T(1) := \lim_{x \nearrow 1} \Phi_T(x) = \lim_{x \nearrow 1} \frac{1 - x^T}{1 - x} = T. \tag{22}$$

### 2.4.3 The case $p = 1$

Since then $P(\tau = 1) = \bar{\alpha} = 1$ we can assume $T = 1$ and by setting $\beta^{-1} := \frac{1-\bar{\alpha}}{R} = 0$ we find Eq. (21) to hold and to coincide with (6) for $p = 1$ and all $T \geq 1$ since $\Phi_T(0) = 1$.

### 2.4.4 The cases $B = M > 0$ and $M = 0$

If the initially bond price $B$ (before receiving the first coupon $c$ at time 0 and paying for insurance) equals the terminal payment $M$, then the coupon $c$ can be interpreted as the sum of the 1-step discounted risk-free interest on $M$ and a risk premium for the loss given default $(1 - X)M$: From Eq. (21) we find for $M > 0$,

$$c = rMR^{-1} + \bar{f}. \tag{23}$$

For $M = 0$ we have $c_0 = 0$ and $c = \delta \geq 0$, so that Eq. (21) becomes

$$B = c\Phi_T(\beta^{-1}). \tag{24}$$

### 2.4.5 The case $c = 0$

Assume $0 \leq p < 1$. Note that we have $c_0 = 0$ iff $rMR^{-1} + \bar{f} \leq \frac{(1-\bar{x})M}{\Phi_{T-1}(\beta^{-1})}$ holds. E.g. for $\bar{x} = 0$, i.e. no recovery, we find $\bar{f} = \bar{\alpha}MR^{-1}$ and $rMR^{-1} + \bar{f} = (1 - \beta^{-1})M \leq \frac{(1-\beta^{-1})M}{1-\beta^{1-T}} = \frac{(1-\bar{x})M}{\Phi_{T-1}(\beta^{-1})}$. For $\bar{x} = 0$, (21) implies for the defaultable term structure

$$B = M\beta^{-T} = MR^{-T}(1 - \bar{\alpha})^T = M(1 + \tilde{r})^{-T}, \tag{25}$$

where $\tilde{r} := \frac{r+\bar{\alpha}}{1-\bar{\alpha}}$. Note that $\tilde{r} > r$ for all $0 < p < 1$ and $\tilde{r}$ can be interpreted as the risk-adjusted interest rate charged to the customer.

### 2.4.6 Uniqueness

We have been working so far under the assumption (5). Extending the functionals $f_t, t \geq 0$, to strictly monotonic functionals defined on all of $L^\infty$, condition (5) is not necessary anymore. Backwards recursively determining $I_t, 1 \leq t \leq \tau \wedge T$, and $(Z_t)_{0 \leq t \leq T}$ from the terminal condition $Z_T = 0$ can still be achieved, but now we have to check, that we do not end up with additional solutions to our replication problem. Uniqueness of the initial bond price $B$ clearly holds for $p \in \{0, 1\}$. Assume $0 < p < 1$ and assume that there exists an initial bond price $\tilde{B} \neq B$ and a stream of extended 1-period insurance contracts with price $\tilde{Y}_t$ and payment $\tilde{I}_{t+1} \in L^\infty, 0 \leq t < \tau \wedge T$, generating a accumulated cash-stream (for fixed $c, M$ as above) $(\tilde{Z}_t)_{0 \leq t \leq T}$ such that $\tilde{Z}_T = 0$ holds. For $\tilde{B} > B$ we find

$\tilde{Z}_0 < Z_0$ and inductively, since $\tilde{I}_{t+1} > I_{t+1}$ we have $\tilde{Y}_t > Y_t$ by strict monotonicity of $f_t$, and we find $\tilde{Z}_t < Z_t$ for all $1 \leq t < \tau \wedge T$. Hence $\tilde{Z}_T < 0$ on $\{\tau = \infty\}$, a contradiction since $P(\tau = \infty) > 0$ by assumption. The case $\tilde{B} < B$ is similar.

### *2.4.7 Continuity properties*

Besides continuity of $B$ under $p \searrow 0$ and $p \nearrow 1$ we also have the following continuity property for $T = 1$: Assume $0 < p < 1$ and $X_\epsilon = X\mathbf{1}_D + \epsilon\mathbf{1}_{\{\tau=1\}\setminus D}$ for $\mathcal{F}_1$-measurable $D \subseteq \{\tau = 1\}$ with $0 \leq P(D) < p$ and $\epsilon \geq 0$. For $T = 1$ a credit with recovery fraction $X_0$ on $\{\tau = 1\}$ generates the same cash-flow as a credit with recovery fraction $X$ and default event $D$. It is now easy to check that the initial bond price corresponding to recovery fraction $X_\epsilon, \epsilon > 0$, and default event $\{\tau = 1\}$ converges to the respective initial bond price for recovery fraction $X$ and default event $D$ as $\epsilon \searrow 0$.

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